

An Order Theoretic Approach to the Banach Contraction Principle in Modular Spaces

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May 6, 2013

Abstract

In the present note, the Banach contraction principle is proved in complete modular spaces via an order theoretic approach.

1 Introduction

A modular formulation of Banach contraction principle was given in [6, 3] under the superfluous assumptions Δ_2 -condition and s -convexity as follows:

Theorem 1 [6] *Let ρ be a function modular satisfying Δ_2 -condition and G be a $\|\cdot\|_\rho$ -closed subset of \mathcal{X}_ρ . If $T : G \rightarrow G$ is a mapping satisfying*

$$\exists c \in [0, 1) : \rho(Tx - Ty) \leq c\rho(x - y) \quad (x, y \in G), \quad (1)$$

and $\sup_n \rho(2T^n x) < \infty$ for some $x \in G$, then T has a fixed point.

Theorem 2 [3] *Let \mathcal{X}_ρ be a ρ -complete modular space. Suppose further that ρ is an s -convex modular satisfying Δ_2 -condition and has the Fatou property. If G is ρ -closed in \mathcal{X}_ρ and $T : G \rightarrow G$ is a mapping satisfying*

$$\rho(c(Tx - Ty)) \leq k^s \rho(x - y) \quad (x, y \in G) \quad (2)$$

for some $c, k \in \mathbb{R}^+$ with $c > \max\{1, k\}$, then T has a fixed point.

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2010 Mathematics Subject Classification. Primary: 47H10, 46E30

Key words: Modular space; Fixed point

In this note, via an order theoretic approach, the modular version of Banach contraction principle is proved. The result also answers, in a more general form, an open question posed in [3].

We commence some basic notions of modular spaces used in theorems above. For more details, the reader is asked to refer to [8, 9]. Some recent works on modular spaces may also be found in [5, 7, 10, 11].

Definition 1 A modular on a linear space \mathcal{X} is a functional $\rho : \mathcal{X} \rightarrow [0, \infty]$ satisfying the conditions:

- 1) $\rho(x) = 0$ if and only if $x = 0$,
- 2) $\rho(x) = \rho(-x)$,
- 3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, for all $x, y \in \mathcal{X}$ and $\alpha, \beta \geq 0$, $\alpha + \beta = 1$.

Then, the vector subspace

$$\mathcal{X}_\rho = \{x \in \mathcal{X} : \rho(\alpha x) \rightarrow 0 \text{ as } \alpha \rightarrow 0\},$$

of \mathcal{X} is called a modular space. If the condition (3) is replaced by

$$\rho(ax + by) \leq a^s \rho(x) + b^s \rho(y)$$

for all $x, y \in \mathcal{X}$ and all $a, b \geq 0$ satisfying $a^s + b^s = 1$, where $s \in (0, 1]$, then the modular ρ is called an s -convex modular on \mathcal{X} .

For every modular space \mathcal{X}_ρ an F -norm $\|\cdot\|_\rho$ can be associated as:

$$\|x\|_\rho = \inf\{t > 0 : \rho(t^{-1}x) \leq t\}, \quad (x \in \mathcal{X}_\rho).$$

Definition 2 Let \mathcal{X}_ρ be a modular space.

- (a) A sequence $\{x_n\}_{n=1}^\infty$ in modular space \mathcal{X}_ρ is called
 - (i) ρ -convergent to $x \in \mathcal{X}_\rho$ if $\rho(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$;
 - (ii) ρ -Cauchy if $\rho(x_m - x_n) \rightarrow 0$ as $m, n \rightarrow \infty$;
- (b) \mathcal{X}_ρ is ρ -complete if every ρ -Cauchy sequence in \mathcal{X}_ρ is ρ -convergent to a point of \mathcal{X}_ρ ;
- (c) A subset C of \mathcal{X}_ρ is said to be ρ -closed if it contains the ρ -limit of all its ρ -convergent sequences;
- (d) ρ is said to satisfy the Δ_2 -condition if for each sequence $\{x_n\}_{n=1}^\infty$ in \mathcal{X}_ρ , $\rho(x_n) \rightarrow 0$, as $n \rightarrow \infty$ implies that $\rho(2x_n) \rightarrow 0$, as $n \rightarrow \infty$;
- (e) The modular ρ has the Fatou property if

$$\rho(x - y) \leq \liminf_{n \rightarrow \infty} \rho(x_n - y_n),$$

whenever $\rho(x_n - x) \rightarrow 0$ and $\rho(y_n - y) \rightarrow 0$ as $n \rightarrow \infty$.

2 Banach Contraction Principle

Theorem 3 *Let \mathcal{X}_ρ be a complete modular space. If ρ has the Fatou property, $T : \mathcal{X}_\rho \rightarrow \mathcal{X}_\rho$ is a mapping satisfying*

$$\exists c \in [0, 1) : \rho(Tx - Ty) \leq c\rho(x - y) \quad (x, y \in \mathcal{X}_\rho), \quad (3)$$

and $\sup_{n \geq 1} \rho(2T^n \omega) < \infty$, for some $\omega \in \mathcal{X}_\rho$, then T has a fixed point.

Proof . Consider the family τ consisting of all subsets \mathcal{M} of $\mathcal{X}_\rho \times [0, \infty)$ such that

$$\exists (b, \beta) \in \mathcal{M}, \forall n \in \mathbb{N} : (T^n b, c^n \beta) \in \mathcal{M},$$

and

$$\forall (x, \alpha), (y, \beta) \in \mathcal{M} : \rho(x - y) \leq |\alpha - \beta|.$$

τ is nonempty. In fact, if ω satisfies $\sup_{n \geq 1} \rho(2T^n \omega) < \infty$, then

$$\sup_{n \geq 1} \rho(\omega - T^n \omega) < \infty,$$

and we may choose $\alpha > 0$ such that $\rho(\omega - T^n \omega) \leq \alpha - c^n \alpha$, for each $n \geq 1$. Then, because of the assumption (3) the set consisting of all $(T^n \omega, c^n \alpha)$, $n \geq 0$ belongs to τ . It is clear, by Zorn's lemma, that the family τ has a maximal element \mathcal{P} with respect to partial order \subseteq in τ . Define the binary relation \preceq in the set \mathcal{P} by

$$(x, \alpha) \preceq (y, \beta) \quad \text{iff} \quad \rho(x - y) \leq \alpha - \beta,$$

where $x, y \in \mathcal{X}_\rho$ and $\alpha, \beta \in [0, \infty)$. Then, (\mathcal{P}, \preceq) is a totally ordered set. We show that \mathcal{P} has a maximum element. If $p \in \mathcal{P}$, then there exist $x_p \in \mathcal{X}_\rho$ and $\alpha_p \in [0, \infty)$ such that $p = (x_p, \alpha_p)$. Note that the net $\{\Lambda_p\}_{p \in \mathcal{P}}$ is increasing, where $\Lambda_p = (x_p, \alpha_p)$. That is,

$$p \preceq q \Rightarrow \rho(x_p - x_q) \leq \alpha_p - \alpha_q. \quad (4)$$

Hence, the net $\{\alpha_p\}_{p \in \Gamma}$ is decreasing in $[0, \infty)$. Replace this net by the sequence $\{\alpha_n\}_{n=1}^\infty$ and assume that $\alpha_n \rightarrow \alpha_0$ as $n \rightarrow \infty$. The inequality (4) shows that the corresponding sequence $\{x_n\}_{n=1}^\infty$ is a ρ -Cauchy sequence in \mathcal{X}_ρ . Suppose that $\{x_n\}_{n=1}^\infty$ is ρ -convergent to $x_0 \in \mathcal{X}_\rho$. Again, from (4) and the Fatou property it follows that

$$\forall p \in \mathcal{P} : \rho(x_p - x_0) \leq \liminf_{m \rightarrow \infty} \rho(x_p - x_m) \leq \alpha_p - \alpha_0.$$

This implies that $(x_p, \alpha_p) \preceq (x_0, \alpha_0)$, for each $p \in \mathcal{P}$ and therefore (x_0, α_0) is the maximum of \mathcal{P} , since \mathcal{P} is maximal.

Now, let (b, β) be an element of \mathcal{P} such that $(T^n b, c^n \beta) \in \mathcal{P}$, for each $n \in \mathbb{N}$. Then, $(T^n b, c^n \beta) \preceq (x_0, \alpha_0)$, for each $n \in \mathbb{N}$ implies that $\alpha_0 = 0$, and therefore $\rho(T^n b - x_0) \rightarrow 0$, as $n \rightarrow \infty$. This also implies that $\rho(T^{n+1} b - T x_0) \rightarrow 0$, as $n \rightarrow \infty$. Hence, $T x_0 = x_0$. \square

Remark. A modular ρ is said to satisfy the Δ_2 -type condition if there exists $\alpha > 0$ such that $\rho(2x) \leq \alpha\rho(x)$ for all $x \in \mathcal{X}_\rho$. The Theorem 3 gets a simpler proof if ρ satisfies the Δ_2 -type condition. To see this, let k be the Δ_2 -type constant of ρ . Without loss of generality, we may suppose that $ck < \frac{1}{2}$. Otherwise, choose the integer n such that $c^n k < \frac{1}{2}$ and then replace T^n by T . Let $x \in \mathcal{X}_\rho$; we show that $\{T^n x\}$ is a ρ -Cauchy sequence. Let $\epsilon > 0$ be given and

$$W_\epsilon = \{(x, y) \in \mathcal{X}_\rho \times \mathcal{X}_\rho : \rho(x - y) < \epsilon\}.$$

There exists $N \in \mathbb{N}$ such that

$$(T^{n-1}x, T^n x) \in W_\epsilon,$$

for each $n > N$. Fix $n > N$. It suffices to show that

$$(T^n x, T^{n+p} x) \in W_\epsilon, \quad (p \in \mathbb{N}). \quad (5)$$

By induction on p , suppose that (5) is valid for some fixed p . Since

$$(T^{n-1}x, T^n x) \in W_\epsilon, \quad (T^n x, T^{n+p} x) \in W_\epsilon$$

we get

$$\rho(T^{n-1}x - T^{n+p}x) \leq ck\rho(T^{n-1}x - T^n x) + ck\rho(T^n x - T^{n+p}x) < \epsilon.$$

Therefore,

$$\rho(T^n x - T^{n+p+1}x) \leq c\rho(T^{n-1}x - T^{n+p}x) < \epsilon.$$

Hence, $\{T^n x\}$ is a ρ -Cauchy sequence and by ρ -completeness must converge. Let $T^n x \rightarrow y \in \mathcal{X}_\rho$, as $n \rightarrow \infty$. From (3), it follows that $T^{n+1}x \rightarrow Ty$, as $n \rightarrow \infty$ and consequently the uniqueness of the ρ -limit implies that $Ty = y$.

The approach given in the proof of Theorem 3 may also be seen in [4]. The idea given in the remark above is, in fact, a simple use of uniform space techniques in fixed point theory (see e.g., [1, 2]).

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